2

Introduction to errors

2.1 The importance of estimating errors

When we measure a physical quantity, we do not expect the value obtained to be exactly equal to the true value. It is important to give some indication of how close the result is likely to be to the true value, that is to say, some indication of the precision or reliability of the measurements. We do this by including with the result an estimate of its error. For example, we might measure the focal length f of a lens and give the final result as

$$f = (256 \pm 2) \text{ mm}. \tag{2.1}$$

By this we mean that we expect the focal length to be somewhere in the range 254 to 258 mm. Equation (2.1) is really a probability statement. It means, not that we are *certain* that the value lies between the limits quoted, but that our measurements indicate that there is a certain *probability* of its doing so. In chapter 3 we shall make this statement more precise.

Estimates of errors are important because without them we cannot draw significant conclusions from the experimental results. Suppose, for example, we wish to find out whether temperature has an effect on the resistance of a coil of wire. The measured values of the resistance are

$$200{\cdot}025\ \Omega \text{ at } 10\ ^\circ\text{C}$$
$$200{\cdot}034\ \Omega \text{ at } 20\ ^\circ\text{C}.$$

Is the difference between these two values significant? Without knowing the errors we cannot say. If, for example, the error in each value of the resistance is $0{\cdot}001\ \Omega$, the difference is significant, whereas if the error is $0{\cdot}010\ \Omega$, then it is not.

Once the result of an experiment has been obtained it goes out into the world and becomes public property. Different people may make use of it in different ways. Some may use it in calculations for a practical end; others may want to compare it with a theoretical prediction. For example, an electrical engineer may want to know the resistivity of copper in order to design a transformer, while a solid state physicist may want

to know the same quantity to test a theory of electrons in metals. Whatever use a person makes of an experimental result, he will want to know whether it is sufficiently precise for his purpose. If he has drawn some conclusions from the result, he will want to know how much confidence to place in them. To answer such questions, an estimate of the error in the result is necessary, and it is the responsibility of the experimenter to provide it.

Now although the experimenter may not be able to foresee all the possible uses of his results, he should be aware of some of them. No experiment should be done in a vacuum - at least not in an intellectual one. If the experiment is being done to test a theory, the experimenter should have some idea how precise the result needs to be in order to provide a useful comparison with the theoretical prediction. So the idea of using error estimates to draw conclusions from the results of an experiment applies also in reverse. That is to say, the purpose of the experiment often determines the error that can be tolerated, and this in turn may have an important influence on the experimental procedure.

It might be thought that every experiment should be done as precisely as possible, but that is an unrealistic point of view. Life is finite, so are the experimenter's resources, and so also, unless he is a genius, is his capacity for taking pains. Therefore it is important to plan and carry out the experiment so that the precision of the final answer is appropriate to the ultimate object of the experiment. Suppose, in the example given at the beginning of this section, that we are only interested in the resistance of the coil because we want to use it as a standard resistance in the temperature range 10 °C to 20 °C, and that the precision required is 1 part in 10 000. A measurement of the resistance with an error of 0·010 Ω would then be quite adequate, and to strive to reduce the error to 0·001 Ω would be a waste of time. On the other hand, to measure the resistance to only 0·05 Ω would be even worse because the measurements would be useless for their purpose.

Just as the final result of an experiment should be obtained to an appropriate degree of precision, so also should the values of the various measured quantities within the experiment. Few experiments are so simple that the final quantity is measured directly. We usually have to measure several primary quantities and bring the results together in order to obtain the quantity required. The errors in the primary quantities determine that in the final result. In general the primary errors contribute different amounts to the final error, and the latter is minimized if the

finite resources of time, apparatus, and patience available are concentrated on reducing those errors that contribute most to the final error.

So we see that the idea of errors is not something of secondary or peripheral interest in an experiment. On the contrary, it is related to the purpose of the experiment, the method of doing it and the significance of the results.

2.2 Systematic and random errors

Errors may be divided into two kinds, systematic and random. A *systematic* error is one which is constant throughout a set of readings.* A *random* error is one which varies and which is equally likely to be positive or negative.

Random errors are always present in an experiment and, in the absence of systematic errors, they cause successive readings to spread about the true value of the quantity - Fig. 2.1a. If in addition a systematic error is present, the readings spread, not about the true value, but about some displaced value - Fig. 2.1b.

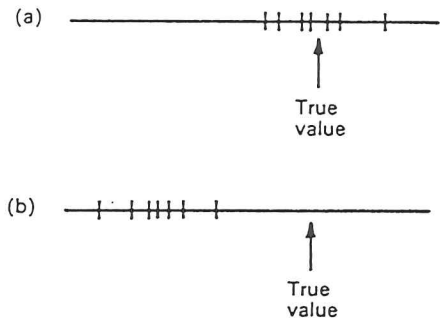

Fig. 2.1. Set of measurements (a) with random errors only and (b) with systematic plus random errors. Each point indicates the result of a measurement.

Suppose that the period of a pendulum is measured by means of a stopclock, and the measurement is repeated many times. Errors in starting and stopping the clock, in estimating the scale divisions, small irregularities in the motion of the pendulum, all these cause variations in the results of successive measurements and may be regarded as random

* This definition is actually too restrictive - some systematic errors are not constant. But in order to give the basic ideas we restrict the discussion here to the simple case. More general cases are considered in chapter 8.

errors. If no other errors are present, some results will be too high and others too low. But if, in addition, the clock is running slow, all the results will be too low. This is a systematic error.

It should be noticed that systematic and random errors are *defined* according to whether they produce a systematic or random effect. So we cannot say that a certain source of error is inherently systematic or random. Returning to the example, suppose that every time we measure the period we use a different clock. Some clocks may be running fast and others slow. But such inaccuracies now produce a random error.

Again, some sources of error may give rise to both systematic and random effects. For example, in operating the clock we might not only start and stop it in a slightly irregular manner in relation to the motion of the pendulum, thus producing a random error, but we might also have a tendency to start it too late and stop it too early, which would give rise to a systematic error.

It is convenient to make a distinction between the words *accurate* and *precise* in the context of errors. Thus a result is said to be *accurate* if it is relatively free from systematic error, and *precise* if the random error is small.

2.3 Systematic errors

Systematic errors often arise because the experimental arrangement is different from that assumed in the theory, and the correction factor which takes account of this difference is ignored. It is easy to give examples of effects that may lead to systematic error. Thermal emfs in a resistance bridge, the resistance of the leads in a platinum thermometer, the effect of the exposed stem in a mercury thermometer, heat losses in a calorimetric experiment, counting losses due to the dead-time in a particle counter are but a few. Another common source of systematic error is the one mentioned earlier - inaccurate apparatus.

Random errors may be detected by repeating the measurements. Furthermore, by taking more and more readings we obtain from the arithmetic mean a value which approaches more and more closely to the true value. Neither of these points is true for a systematic error. Repeated measurements with the same apparatus neither reveal nor do they eliminate a systematic error. For this reason systematic errors are potentially more dangerous than random errors. If large random errors are present in an experiment, they will manifest themselves in a large value of the final quoted error. Thus everyone is aware of the imprecision of the

result, and no harm is done - except possibly to the ego of the experimenter when no one takes notice of his results. However, the concealed presence of a systematic error may lead to an apparently reliable result, given with a small estimated error, which is in fact seriously wrong.

A classic example of this was provided by Millikan's oil-drop experiment to measure e, the elementary charge. In this experiment it is necessary to know the viscosity of air. The value used by Millikan was too low, and as a result the value he obtained for e was

$$e = (1{\cdot}591 \pm 0{\cdot}002) \times 10^{-19}\ \text{C}.$$

This may be compared with the present value (Cohen and Taylor 1973)

$$e = (1{\cdot}602\,189 \pm 0{\cdot}000\,005) \times 10^{-19}\ \text{C}.$$

Up till 1930, the values of several other atomic constants, such as the Planck constant and the Avogadro constant, were based on Millikan's value for e and were consequently in error by more than $\frac{1}{2}\%$.

Random errors may be estimated by statistical methods, which are discussed in the next two chapters. Systematic errors do not lend themselves to any such clear-cut treatment. Your safest course is to regard them as effects to be discovered and eliminated. There is no general rule for doing this. It is a case of thinking about the particular method of doing an experiment and of always being suspicious of the apparatus. We shall try to point out common sources of systematic error in this book, but in this matter there is no substitute for experience.

3

Treatment of a single variable

3.1 Introduction

Suppose we make a set of measurements, free from systematic error, of the same quantity. The individual values x_1, x_2, etc., vary owing to random errors, and the mean value $\bar{x}$ (i.e. the arithmetic average) is taken as the best value of the quantity. But, unless we are lucky, $\bar{x}$ will not be equal to the true value of the quantity, which we denote by X. The question we are going to consider is how close we expect $\bar{x}$ to be to X. Of course we do not know the *actual* error in $\bar{x}$. If we did, we would correct $\bar{x}$ by the required amount and get the right value X. The most we can do is to say that there is a certain probability that X lies within a certain range centred on $\bar{x}$. The problem then is to calculate this range for some specified probability.

A clue to how we should proceed is provided by the results shown in Fig. 3.1. On the whole, for the results in Fig. 3.1a we would expect X to be fairly close to $\bar{x}$; whereas for those in Fig. 3.1b we would not be greatly surprised if there were quite a large difference. In other words, the larger the spread in the results, the larger we would expect the error

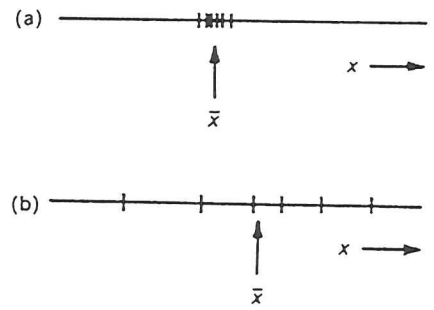

Fig. 3.1. Results of successive measurements of the same quantity. The mean $\bar{x}$ is expected to be closer to the true value for set (a) than for set (b).

Table 3.1. *Measurements of the resistance R of a coil*

R/Ω	R/Ω
4·615	4·613
4·638	4·623
4·597	4·659
4·634	4·623

in $\bar{x}$ to be. The whole of the present chapter is concerned with putting this qualitative statement on a firm quantitative basis. We assume throughout that no systematic errors are present.

3.2 Set of measurements

Denote the values of n successive measurements of the same quantity by

$$x_1, x_2, \ldots, x_n. \tag{3.1}$$

The number n is not necessarily large and in a typical experiment might be in the range 5 to 10. The mean is

$$\bar{x} = \frac{1}{n}\sum x_i. \tag{3.2}$$

(Whenever the symbol $\sum$ appears in the present chapter, the summation is to be taken from $i = 1$ to $i = n$.)

To fix our ideas let us consider a specific experiment in which the resistance of a coil is measured on a bridge. The measurement is made $n = 8$ times. The results are listed in Table 3.1. The mean of these values is 4·625 Ω. We require a quantity that gives a measure of the spread in the 8 values, from which we shall estimate the error in the mean. To define such a quantity we need to introduce the idea of a *distribution* – one of the basic concepts in the theory of statistics.

3.3 Distribution of measurements

(*a*) ***Introduction.*** Although we have only n actual measurements, we *imagine* that we go on making the measurements so that we end up with a very large number N. We may suppose N is say 10 000 000. (Since we are not actually making the measurements, expense is no object.) We call this hypothetical set of a very large number of readings a *distribution.*

The basic idea to which we shall constantly return is that *our actual set of n measurements is a random sample taken from the distribution of N measurements.*

We may represent any set of measurements by a *histogram.* This is true whether it be a set of a small number n of measurements or a distribution of a large number N. To construct a histogram we simply divide the range of measured values into a set of equal intervals and count how many times a measurement occurs in each interval. The width of the intervals is arbitrary and is chosen for convenience. Figure 3.2 shows a histogram for the measurements in Table 3.1.

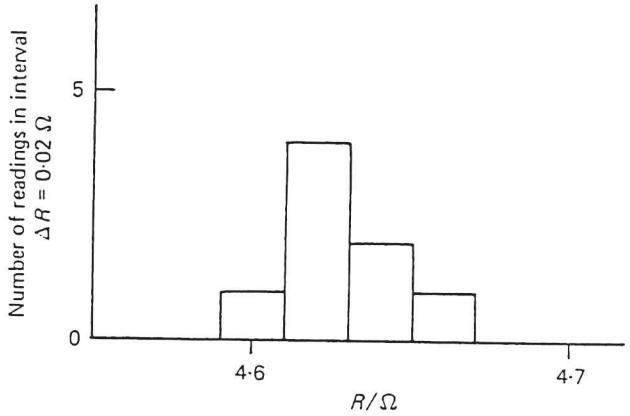

Fig. 3.2. Histogram of the readings given in Table 3.1.

This histogram has a jagged appearance because it represents only a few values. However, suppose we represent a distribution in this way. The number of measurements N is so large that we may make the width of the intervals very small and - provided the measuring instrument gives sufficiently fine readings - still have an appreciable number of readings in each interval. If we plot, instead of the histogram itself, the *fraction* of the N readings in each interval as a function of the value of the measurement, we shall get a smooth curve. We may then define a function $f(x)$, known as the *distribution function*, whose significance is that $f(x)\,dx$ is the fraction of the N readings that lie in the interval x to $x+dx$. In other words, $f(x)dx$ is the *probability* that a single measurement taken at random from the distribution will lie in the interval x to $x+dx$. We shall not specify the exact form of $f(x)$ at this stage but we expect a typical distribution function to look something like that shown in Fig. 3.3.

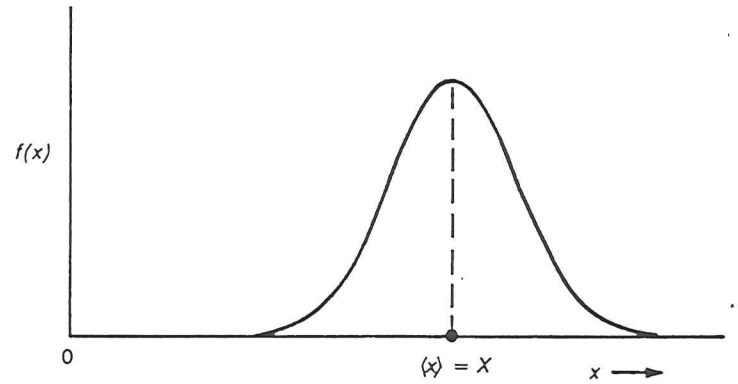

Fig. 3.3. Typical function for distribution of measurements.

From its definition $f(x)$ satisfies the relation

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \tag{3.3}$$

Notice the infinite limits in the integral. We do not expect any measurements with values greatly different from the true value X in an actual experiment. In particular, many quantities are of a nature such that negative values are impossible. Therefore any function $f(x)$ that we use to represent a distribution must become very small as the difference between x and X becomes large. For such functions no difficulty arises from the infinite limits, and they are taken for mathematical convenience.

We shall use the symbol $\langle\,\rangle$ to denote an average over all the measurements in the distribution. An important average is the *mean* of the distribution

$$\langle x\rangle = \int_{-\infty}^{\infty} x f(x)\,dx. \tag{3.4}$$

Since the number of measurements in the distribution is very large, and they are free from systematic error, $\langle x\rangle$ may be taken as equal to the true value X.

(*b*) ***Standard error in a single observation.*** The error in a measurement with value x is

$$e = x - X. \tag{3.5}$$

The rms (root-mean-square) value of e for all the measurements in the distribution is denoted by σ and is known as the *standard deviation of*

*the distribution.** Thus σ is defined by the equation

$$\sigma^2 = \langle e^2 \rangle = \int_{-\infty}^{\infty} (x - X)^2 f(x)\, dx. \tag{3.6}$$

The standard deviation is a measure of the spread of the distribution, i.e. of the scatter of the measurements. A distribution representing a precise set of measurements will be highly peaked near $x = X$ and will have a small value for σ; while one representing an imprecise set will have a large scatter about X and a large value for σ (Fig. 3.4). We take the quantity σ as our measure of the error in a single observation, and it is therefore also referred to as the *standard error in a single observation.*

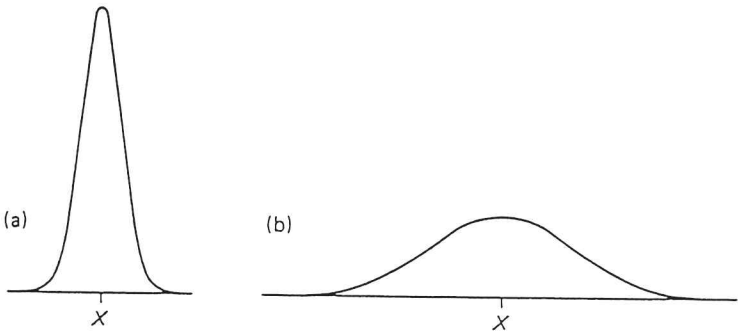

Fig. 3.4. Distribution function $f(x)$ for (a) a precise set of measurements (small value of σ), and (b) an imprecise set of measurements (large value of σ). Note that the area under the two curves is the same because both functions satisfy the relation (3.3).

(*c*) ***Standard error in the mean.*** We now proceed to define a quantity that specifies the error in the mean of a set of n measurements.

Let us go back to the set of 8 measurements given in Table 3.1. We have said that they are to be regarded as a random sample taken from the distribution of single measurements. Imagine the distribution to be represented by a bin containing N balls, each of which is labelled with the value of a single measurement. The set of measurements in Table 3.1 may be regarded as the result of drawing 8 balls at random from the bin. Suppose now we have a second bin, initially empty, and a supply of blank balls. We look at our 8 measurements, calculate their mean, write the answer on a blank ball and put it in the second bin. We put the set of 8 single-measurement balls back into the first bin, stir the contents,

* The quantity σ^2 is known as the *variance* of the distribution.

and take out, at random, a second set of 8 balls. We again calculate the mean of the 8 values, write it on a blank ball, and put it in the second bin. We continue the process a large number of times, always drawing the same number, 8, of balls from the first bin. We end up with a large number of values in the second bin, which represents another distribution, namely a distribution of means of 8 measurements.

We denote the standard deviation of this distribution by σ_m. We take it as our measure of the error in the mean of a single set of n measurements, and it is therefore known as the *standard error in the mean.*

To sum up, σ is the standard deviation of the distribution of single measurements, and σ_m is the standard deviation of the distribution of the means of sets of measurements, each set containing the same number n of single measurements. σ represents the error in a single measurement, and σ_m represents the error in the mean of n measurements.

(*d*) ***Relation between σ and σ_m.*** There is a simple relation between σ and σ_m which we now prove. Consider one set of n measurements $x_1, \ldots, x_n$. The error in the ith reading is

$$e_i = x_i - X, \tag{3.7}$$

where X is the true value of the quantity, which is of course unknown. The error in the mean is

$$E = \bar{x} - X = \left(\frac{1}{n}\sum x_i\right) - X = \frac{1}{n}\sum (x_i - X) = \frac{1}{n}\sum e_i. \tag{3.8}$$

Therefore

$$E^2 = \frac{1}{n^2}\sum e_i^2 + \frac{1}{n^2}\sum_{\substack{i}}\sum_{\substack{j \\ i \neq j}} e_i e_j. \tag{3.9}$$

This is for a single set of n measurements. Now imagine that, as before, we take a large number of sets, each set consisting of the same number n of single measurements. Each set will have its own set of values for $e_1, \ldots, e_n$, and its own value of E. Equation (3.9) will be true for each set. We add the equations for all the sets and then divide by the number of sets, that is to say, we average (3.9) over all the sets. The average of $\sum e_i^2$ is $n\langle e^2 \rangle$. The average of each term in the double sum is zero, because the errors e_i and e_j are independent, and the average of each is zero. We therefore arrive at the result

$$\langle E^2 \rangle = \frac{1}{n}\langle e^2 \rangle. \tag{3.10}$$

By definition

$$\sigma_m^2 = \langle E^2 \rangle \quad \text{and} \quad \sigma^2 = \langle e^2 \rangle. \tag{3.11}$$

Equation (3.10) thus becomes

$$\sigma_m = \frac{\sigma}{\sqrt{n}}, \tag{3.12}$$

i.e. *the standard error in the mean of n observations is* $1/\sqrt{n}$ *times the standard error in a single observation.*

The value of σ depends only on the precision of the individual measurements and is independent of their number; whereas the value of σ_m can be reduced by increasing n. However, since σ_m decreases only as $1/\sqrt{n}$, it becomes more and more unprofitable to take readings of the same quantity. Rather we should try to reduce σ_m by reducing σ, i.e. by taking a more precise set of readings in the first place.*

3.4 Estimation of σ and σ_m

(*a*) ***Standard method.*** We have defined the quantities σ and σ_m that we are going to take as our measures of the error in a single measurement and in the mean. It remains to show how we can calculate, or more correctly how we can best estimate, them from the actual measurements. We need to estimate only one of them, because we can then use (3.12) to obtain the other.

The best estimate of σ is $[(1/n)\sum e_i^2]^{\frac{1}{2}}$, but the errors e_i come from the true value X and hence are not known. A way round the difficulty is provided by working in terms of *residuals.*

The residual d_i of the ith measurement is defined by

$$d_i = x_i - \bar{x}. \tag{3.13}$$

Unlike the error, the residual is a known quantity. We denote the rms value of the n residuals by s, i.e.

$$s^2 = \frac{1}{n}\sum d_i^2. \tag{3.14}$$

The quantity s is called the *standard deviation of the sample.* From (3.7) and (3.8)

$$x_i - \bar{x} = e_i - E. \tag{3.15}$$

* A splendid example of this approach is to be found in the experiment to measure g precisely - see section 7.4 (d).

Therefore

$$\begin{aligned} s^2 &= \frac{1}{n}\sum (x_i - \bar{x})^2 = \frac{1}{n}\sum (e_i - E)^2 \\ &= \frac{1}{n}\sum e_i^2 - 2E\frac{1}{n}\sum e_i + E^2 \\ &= \frac{1}{n}\sum e_i^2 - E^2. \end{aligned} \tag{3.16}$$

This is for one set of n measurements. As before, we take the average of this equation over a large number of sets in the distribution and obtain the result

$$\langle s^2 \rangle = \sigma^2 - \sigma_m^2. \tag{3.17}$$

From (3.12) and (3.17) we have

$$\sigma^2 = \frac{n}{n-1}\langle s^2 \rangle, \tag{3.18}$$

and

$$\sigma_m^2 = \frac{1}{n-1}\langle s^2 \rangle. \tag{3.19}$$

The quantity $\langle s^2 \rangle$ is not known. Our best estimate of it is s^2, obtained by evaluating (3.14). Substituting this value and taking the square root gives

$$\sigma \approx \left(\frac{n}{n-1}\right)^{\frac{1}{2}} s, \tag{3.20}$$

$$\sigma_m \approx \left(\frac{1}{n-1}\right)^{\frac{1}{2}} s. \tag{3.21}$$

We thus have estimates of σ and σ_m in terms of quantities that are known.*

(*b*) ***Worked example.*** As an example we show how σ and σ_m are estimated for the set of measurements in Table 3.1, which are listed again in the first column of Table 3.2. The first step is to calculate the mean, which is 4·625 Ω. From the mean we calculate the residual of each measurement. For example, the residual of the first measurement is

$$d_1 = (4{\cdot}615 - 4{\cdot}625)\ \Omega = -10\ \text{m}\Omega. \tag{3.22}$$

The residuals and their squares are listed in the second and third columns

* The symbol $\approx$ signifies that (3.20) and (3.21) are not strictly equations. The values of the right-hand sides depend on the particular set of n measurements and are not in general exactly equal to σ and σ_m - see section 3.7.

Table 3.2. *Estimation of σ and σ_m for the measurements in Table 3.1*

Resistance R/Ω	Residual $d/\mathrm{m}\Omega$	$(d/\mathrm{m}\Omega)^2$
4·615	−10	100
4·638	13	169
4·597	−28	784
4·634	9	81
4·613	−12	144
4·623	−2	4
4·659	34	1156
4·623	−2	4
mean = 4·625		sum = 2442

of Table 3.2. Then

$$s^2=\frac{1}{n}\sum d_i^2=\frac{2442}{8}\times 10^{-6}\,\Omega^2, \qquad s=0{\cdot}017\,\Omega, \tag{3.23}$$

$$\sigma\approx\left[\frac{n}{n-1}\right]^{\frac{1}{2}}s=\left(\frac{8}{7}\right)^{\frac{1}{2}}\times 0{\cdot}017=0{\cdot}019\,\Omega, \tag{3.24}$$

$$\sigma_m=\frac{\sigma}{\sqrt{n}}\approx\frac{0{\cdot}019}{\sqrt{8}}=0{\cdot}007\,\Omega. \tag{3.25}$$

The result of a set of measurements is quoted as $\bar{x}\pm\sigma_m$. So in the present case our best estimate of the resistance of the coil is

$$R=4{\cdot}625\pm 0{\cdot}007\,\Omega. \tag{3.26}$$

(*c*) ***Programmed calculator.*** Calculators, programmed to calculate σ, use the standard method, but they do not evaluate s from (3.14) because that requires the value of $\bar{x}$, which is not known until all the numbers are fed in. However, there is an alternative expression for s that avoids the difficulty. From (3.2), (3.13) and (3.14) we have

$$\begin{aligned}s^2&=\frac{1}{n}\sum(x_i-\bar{x})^2\\&=\frac{1}{n}\left[\sum x_i^2-2\bar{x}\sum x_i+n\bar{x}^2\right]\\&=\frac{1}{n}\sum x_i^2-\left(\frac{1}{n}\sum x_i\right)^2.\end{aligned} \tag{3.27}$$

Combining this with (3.20) gives

$$\sigma\approx\left(\frac{1}{n-1}\right)^{\frac{1}{2}}\left[\sum x_i^2-\frac{1}{n}(\sum x_i)^2\right]^{\frac{1}{2}}. \tag{3.28}$$

This is the expression used by a calculator programmed to evaluate standard deviations. As the numbers x_i are fed in, the calculator accumulates the values of $\sum x_i^2$ and $\sum x_i$. It then uses (3.2) and (3.28) to evaluate $\bar{x}$ and σ. Similarly

$$\sigma_m\approx\left[\frac{1}{n(n-1)}\right]^{\frac{1}{2}}\left[\sum x_i^2-\frac{1}{n}(\sum x_i)^2\right]^{\frac{1}{2}}. \tag{3.29}$$

(*d*) ***Deviations from a general value of x.*** Before leaving this section we prove one further result. Suppose that instead of taking the deviations of the readings from the mean $\bar{x}$, as we have done so far, we take the deviations from a different value of x. Denote by $S(x)$ the rms deviation of the readings taken from x, so that

$$[S(x)]^2=\frac{1}{n}\sum(x_i-x)^2. \tag{3.30}$$

Combining this equation with (3.27) we have

$$\begin{aligned}[S(x)]^2-s^2&=\frac{1}{n}\sum[(x_i-x)^2-(x_i-\bar{x})^2]\\&=\frac{1}{n}\sum(x^2-2x_ix+2x_i\bar{x}-\bar{x}^2)\\&=x^2-2\bar{x}x+2\bar{x}^2-\bar{x}^2=(x-\bar{x})^2,\end{aligned}$$

i.e.

$$[S(x)]^2=s^2+(x-\bar{x})^2. \tag{3.31}$$

This demonstrates an important result. For a given set of readings the sum of the squares of the deviations is a minimum when the deviations are taken from the mean of the set. That is the reason why s^2 is not an unbiased estimate of σ^2. It is slightly too small, as (3.18) shows.

3.5 The Gaussian distribution

We have not yet specified the exact form of the distribution function $f(x)$. The results derived so far are therefore independent of the distribution. However, to make further progress we need a specific function, and the one we shall take is

$$f(x)=\frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}\exp[-(x-X)^2/2\sigma^2]. \tag{3.32}$$

This function, specified by the two constants X and σ, is known as a *Gaussian*, and the distribution as a *Gaussian* or *normal* distribution. Its shape is shown in Fig. 3.5.

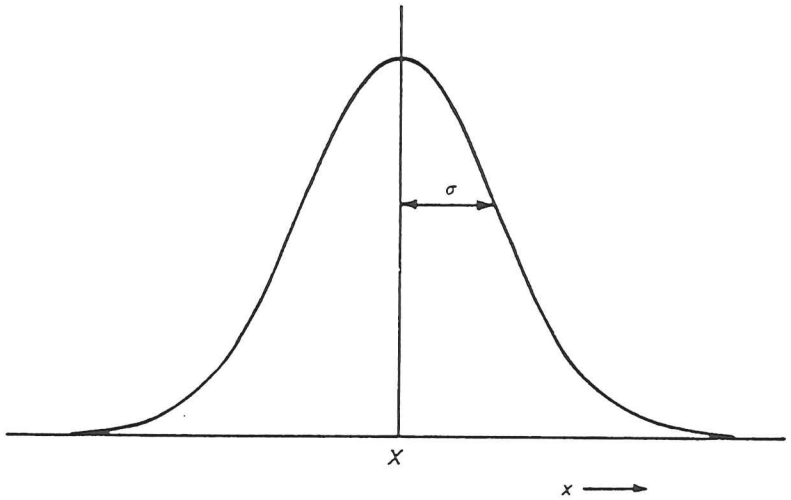

Fig. 3.5. The Gaussian distribution function. The points of inflexion are at $x = X \pm \sigma$.

Later in the chapter we mention the reasons for choosing the Gaussian, but at this stage we merely note that the function

(i) is symmetric about X,
(ii) has its maximum value at X,
(iii) tends rapidly to zero as $|x-X|$ becomes large compared with σ.

Clearly these are reasonable properties for a function representing a distribution of measurements containing only random errors.

We show below that the constant σ in (3.32) is in fact the standard deviation of the function – hence the choice of the symbol. The purpose of the multiplying factor

$$\frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}$$

is to make $f(x)$ satisfy (3.3). We can see that it does so from the value of the first integral in Table 3.3. (The results in the table are proved in Appendix B.) Put $X = 0$, which does not affect the argument. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2)\,dx = 1. \qquad (3.33)$$

Table 3.3. *Some useful integrals for the Gaussian distribution*

$$\int_{-\infty}^{\infty} \exp(-x^2/2\sigma^2)\,dx = \sqrt{(2\pi)}\sigma$$

$$\int_{-\infty}^{\infty} x^2 \exp(-x^2/2\sigma^2)\,dx = \sqrt{(2\pi)}\sigma^3$$

$$\int_{-\infty}^{\infty} x^4 \exp(-x^2/2\sigma^2)\,dx = 3\sqrt{(2\pi)}\sigma^5$$

The standard deviation of the function in (3.32) is obtained from the second integral in Table 3.3. We continue with $X = 0$. By definition

$$\begin{aligned}(\text{standard deviation})^2 &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\ &= \frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}\int_{-\infty}^{\infty} x^2 \exp(-x^2/2\sigma^2)\,dx \\ &= \sigma^2. \qquad (3.34)\end{aligned}$$

It is readily verified that the points of inflexion of the function $\exp(-x^2/2\sigma^2)$ occur at $x = \pm\sigma$, a convenient result for relating the standard deviation to the shape of a Gaussian.

3.6 The integral function

Suppose we have a symmetric distribution represented by a function $f(x)$ for which $X = 0$. We can ask what fraction $\phi(x)$ of the measurements lie within the range $-x$ to x. Since $f(x)\,dx$ is by definition the fraction of readings between x and $x+dx$, $\phi(x)$ is given by

$$\phi(x) = \int_{-x}^{x} f(y)\,dy. \qquad (3.35)$$

We call $\phi(x)$ the *integral function* of the distribution. It is equal to the shaded area in Fig. 3.6 divided by the total area under the curve.

For a Gaussian distribution with standard deviation σ

$$\phi(x) = \frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}\int_{-x}^{x} \exp(-y^2/2\sigma^2)\,dy. \qquad (3.36)$$

The function $\phi(x)$ depends on σ. It is convenient to have one table of

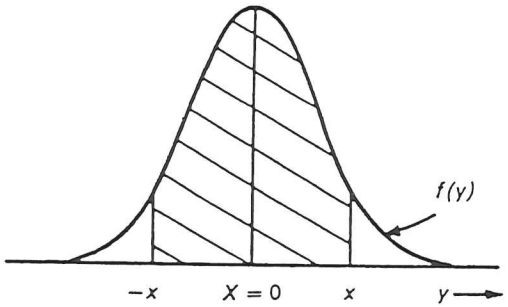

Fig. 3.6. $\phi(x)$, the fraction of measurements within $\pm x$, is the ratio of the shaded area to the total area under the distribution function $f(y)$.

values that can be used for all values of σ. The variable is therefore changed to $t = y/\sigma$. Put $z = x/\sigma$. Then

$$\phi(z) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^z \exp(-t^2/2)\, dt. \qquad (3.37)$$

The function $\phi(z)$ must be evaluated by numerical methods. It is tabulated in Appendix A and plotted in Fig. 3.7.

A few selected values of $\phi(z)$ are given in Table 3.4. The numbers in the third column are worth remembering. We see that about two out of

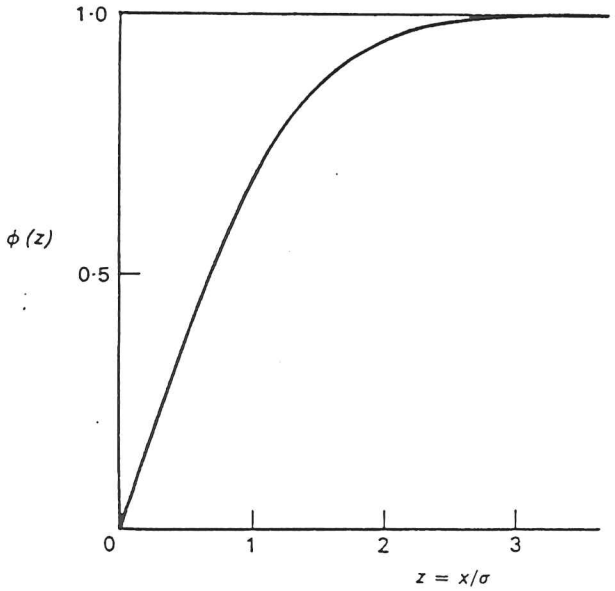

Fig. 3.7. The integral function $\phi(z)$ for the Gaussian distribution.

Table 3.4. *Values of the Gaussian integral function*

$z = x/\sigma$	$\phi(z)$	Approximate fraction of readings outside z value
0	0	1 out of 1
1	0·683	3
2	0·954	20
3	0·9973	400
4	0·99994	16 000

three observations lie within $\pm\sigma$. About 1 in 20 of the observations lie outside 2σ, and about 1 in 400 outside 3σ.

These results provide a quantitative basis for the statement that σ is a measure of the spread of the observations. They also provide a check that σ has been estimated correctly. For a set of readings with mean $\bar{x}$, roughly two-thirds of the readings should lie in the range $\bar{x} \pm \sigma$. We can also apply the results to the interpretation of σ_m, remembering that σ_m is the standard deviation of the distribution of means of which $\bar{x}$ is a member. Thus, when we quote the result of the measurements as $\bar{x} \pm \sigma_m$, the implication is that, in the absence of systematic error, the probability that the true value of the quantity lies in the quoted range is roughly two-thirds.

In addition to σ, another quantity sometimes used to specify the error in the measurements is the so-called *probable error*. It is defined as the value of x such that one-half of the readings lie within x of the true value. For the Gaussian distribution the probable error is equal to $0{\cdot}67\sigma$. There is little point in having two similar measures of the error, and it is obviously convenient to settle on one of them. Though its significance is easy to appreciate, the probable error is not a very fundamental quantity, and the error commonly quoted nowadays is the standard error. We use it throughout the present book and only mention the probable error because it may be encountered in older books and papers.

3.7 The error in the error

It was stated in section 3.4 that the best estimate of $\langle s^2 \rangle$ is provided by s^2. However, s^2 is just the value we happen to get from the particular set of n readings. It is of interest to know how s^2 varies from one set to

another. The error in s^2 is

$$u = s^2 - \langle s^2 \rangle. \tag{3.38}$$

In Appendix C we show that for a Gaussian distribution the mean value of u^2 (taken from a large number of sets of n readings) is $[2/(n-1)]\langle s^2\rangle^2$. Hence the fractional standard deviation of s^2 is $[2/(n-1)]^{\frac{1}{2}}$, and, provided n is fairly large, the fractional standard deviation of s is approximately half this value, i.e. $1/(2n-2)^{\frac{1}{2}}$.

The quantity $1/(2n-2)^{\frac{1}{2}}$ is plotted against n in Fig. 3.8. The result provides a salutary warning against elaborate calculations of errors. Notice for example that for $n = 9$, a not insignificant number of readings, the error estimate is only good to 1 part in 4.

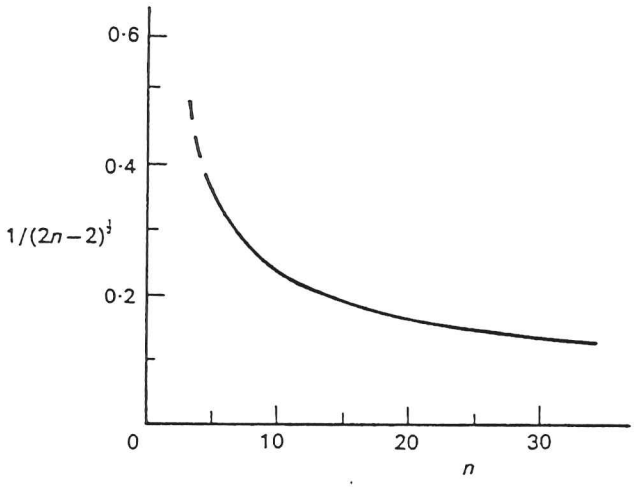

Fig. 3.8. $1/(2n-2)^{\frac{1}{2}}$, representing the fractional standard deviation of s, plotted against n, the number of measurements.

3.8 Range method of estimating σ and σ_m

The standard method of estimating σ and σ_m (section 3.4) is via s, the standard deviation of the sample. However, if you do not have a calculator programmed to calculate standard deviations, the expressions for s - (3.14) and (3.27) - are tedious to evaluate. Fortunately there are other methods of estimating σ and σ_m, which are easier to use, and are almost as reliable as the standard method. The simplest of these is the *range method.* It assumes that the distribution function is Gaussian.

Let r be the difference between the largest and smallest value in a set of n readings. Then an estimate of σ is given by the formula

$$\sigma \approx \frac{r}{v_n}, \tag{3.39}$$

where v_n is a number which may be calculated from statistical theory (Guest 1961, p. 44). It turns out that for n up to about 12, an adequate approximation for v_n is $\sqrt{n}$. So the method reduces to the simple formula

$$\sigma \approx \frac{r}{\sqrt{n}}. \tag{3.40}$$

Remember that σ is independent of n. Formula (3.40) therefore implies that the average value of the range r of a set of n readings is proportional to $\sqrt{n}$.

Applying the range method to the set of readings in Table 3.1 we have

$$r = 4{\cdot}659 - 4{\cdot}597 = 0{\cdot}062\ \Omega. \tag{3.41}$$

Therefore

$$\sigma \approx \frac{0{\cdot}062}{\sqrt{8}} = 0{\cdot}022\ \Omega, \tag{3.42}$$

which may be compared with $\sigma \approx 0{\cdot}019\ \Omega$ given by the standard method.

If we combine (3.40) with the relation

$$\sigma_m = \frac{\sigma}{\sqrt{n}}, \tag{3.12}$$

we have

$$\sigma_m \approx \frac{r}{n}, \tag{3.43}$$

which is a particularly simple formula for estimating σ_m. The formula does *not*, of course, imply that σ_m is proportional to $1/n$. Since r is proportional to $\sqrt{n}$, the formula implies correctly that σ_m is proportional to $1/\sqrt{n}$.

For values of n up to 12, the fractional standard deviations of the estimates of σ given by the range method are only about 10% higher than those given by the standard method. But for n greater than 12 the range estimates become increasingly unreliable. Also $\sqrt{n}$ becomes an increasingly poor approximation for v_n. ($\sqrt{n}$ becomes greater than v_n, so the range method tends to underestimate σ.) However, for values of n up to about 12, the range method of estimating σ and σ_m is quite adequate for most purposes.

3.9 Discussion of the Gaussian distribution

Much has been written about the validity of the Gaussian distribution in the theory of errors. Perhaps the best known comment is that experimenters believe in it because they think it can be proved by mathematics, and mathematicians because they think it has been established by experiment. However, the Gaussian distribution does have some theoretical basis. For example, it may be shown to follow from the assumption that each observation is the result of a large number of independent errors, small but discrete, roughly equal in magnitude, and equally likely to be positive or negative.

The assumption of a Gaussian distribution is related to taking the mean of a set of readings as the best value of the measured quantity. The word 'best' in this context is defined as follows. Suppose the distribution function has the form $f(x-X)$, where as usual X is the true value of the quantity. Let ε be the smallest quantity to which the measuring instrument is sensitive. (We suppose ε to be small - its actual value does not affect the argument.) The probability that we shall obtain the values $x_1, x_2, \ldots, x_n$ when we make n measurements is

$$f(x_1-X)f(x_2-X)\ldots f(x_n-X)\varepsilon^n. \tag{3.44}$$

The best value of X is defined to be the one which when inserted into (3.44) makes the quantity a maximum, i.e. it is the value which maximizes the probability of getting the particular set of measured values. Now it can readily be proved that if $f(x-X)$ is a Gaussian, the best value of X is the mean of x_1 to x_n, and conversely that if the best value of X is the mean, then the distribution function is a Gaussian (Whittaker and Robinson 1944, p. 218.)

The Gaussian is the only distribution we shall use, but this should not be taken to imply that all experimental distributions in physics are of this form. Phenomena in which a random process gives rise to discrete measured values - for example, the counting of particles in atomic and nuclear physics - follow the Poisson distribution. This is discussed in Appendix D.

The results we have derived using the Gaussian distribution are in fact rather insensitive to the precise form of the distribution. This applies to the results in sections 3.6, 3.7 and 3.8. We have seen - section 3.7 - that, quite apart from the question of the form of the distribution, the values obtained for the errors are, in the majority of cases, fairly crude estimates. Their uncertainties completely swamp effects due to the distribution being slightly non-Gaussian.

The main thing is to have some distribution which is (a) reasonable and (b) easy to handle algebraically. In most cases the Gaussian distribution fulfils both conditions very nicely. So, unless the measurements provide clear evidence to the contrary, we assume that the distribution is Gaussian and use the formulae based on it. The one *common* instance of non-Gaussian distribution is when the measurements are discrete, being readings of an instrument to the nearest scale division. This situation is discussed in chapter 5.

Summary of symbols, nomenclature, and important formulae

A. *Set of n measurements*

Quantities that are known

measured values	$x_1, x_2, \ldots, x_n$
mean	$\bar{x} = \frac{1}{n}\sum x_i$
residual for ith reading	$d_i = x_i - \bar{x}$
standard deviation of sample	$s = \left(\frac{1}{n}\sum d_i^2\right)^{\frac{1}{2}}$

Quantities that are not known

true value	X
error in ith reading	$e_i = x_i - X$
error in mean	$E = \bar{x} - X$

B. *Distributions*

Distribution of single measurements

standard error $\sigma = \langle e^2 \rangle^{\frac{1}{2}}$

Distribution of means of n measurements

standard error $\sigma_m = \langle E^2 \rangle^{\frac{1}{2}}$

$\langle\ \rangle$ denotes the average over the distribution.

C. *Important relations*

$$\sigma_m = \frac{\sigma}{\sqrt{n}}$$

$$\sigma^2 = \frac{n}{n-1}\langle s^2 \rangle$$

$$\sigma_m^2 = \frac{1}{n-1}\langle s^2 \rangle$$

D. *Formulae for estimating σ and σ_m*

Standard method

$$\sigma \approx \left[\frac{\sum d_i^2}{n-1}\right]^{\frac{1}{2}} = \left[\frac{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}{n-1}\right]^{\frac{1}{2}}$$

$$\sigma_m \approx \left[\frac{\sum d_i^2}{n(n-1)}\right]^{\frac{1}{2}} = \left[\frac{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}{n(n-1)}\right]^{\frac{1}{2}}$$

Range method (suitable for $2 \leqslant n \leqslant 12$)

r = difference between largest and smallest reading in set

$$\sigma \approx \frac{r}{\sqrt{n}} \qquad \sigma_m \approx \frac{r}{n}$$

E. *Gaussian distribution*

$$f(x) = \frac{1}{\sqrt{(2\pi)}}\frac{1}{\sigma}\exp\left[-(x-X)^2/2\sigma^2\right].$$

Put $X = 0$. The fraction of readings between x and $x + dx$ is $f(z)\,dz$, where

$$f(z) = \frac{1}{\sqrt{(2\pi)}}\exp(-z^2/2), \qquad z = \frac{x}{\sigma}.$$

The fraction of readings between $-x$ and $+x$ is

$$\phi(z) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^z \exp(-t^2/2)\,dt.$$

$f(z)$ and $\phi(z)$ are tabulated in Appendix A.

Exercises

3.1 *Dartboard experiment.* The following experiment is designed to illustrate some of the basic results in the theory of random errors. Inserting numbers into the equations to see how they work in practice is a considerable aid to understanding the ideas behind them.

The measured values of some hypothetical quantity are simulated by throwing darts at the central vertical column of the target shown in Fig. 3.9. Recommended dimensions for the target are 12 mm for the width of each column and 0·6 m for the length. The target should be mounted vertically - or sloping backwards a few degrees - on a soft board about 0·5 m wide and 0·6 m long. Throw the darts standing behind a line 2 m from the target.

Five darts are used in the exercise and they are thrown in succession to give a set of 5 readings. Try to throw each one at the central column without being influenced by the positions of previous throws in the set. Throw 30 sets of 5 and record the number of the column in which each dart falls. If a dart falls out of the board before it is recorded, throw it again. If it falls outside the columns ±10, measure its position and record the equivalent column. You will find it easier to do the subsequent calculations if you record the results by drawing a set of columns to match those of the target and marking each throw as a tick in the appropriate column. Record each set of 5 throws on a separate line.

The total of 150 readings is taken to represent a distribution of single measurements. (In chapter 3 we said that a distribution is a collection of a very large number of readings. Although 150 is not a very large number it is adequate for the present purpose. The same comment applies to the

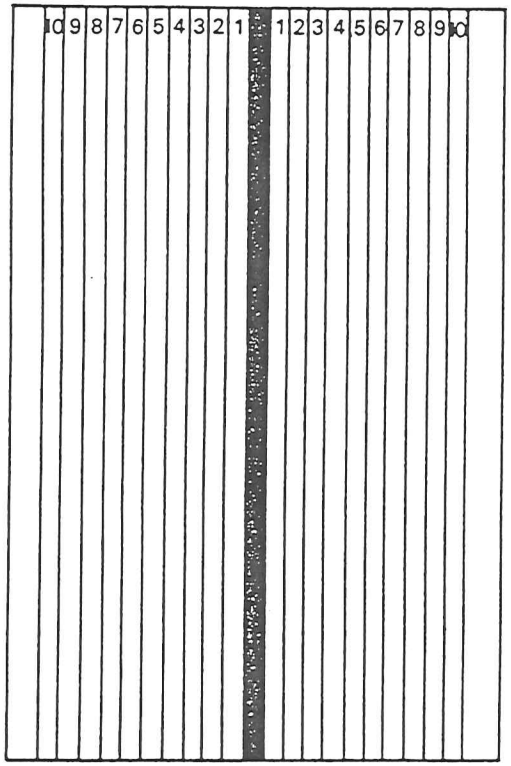

Fig. 3.9. Target for dartboard experiment.

distribution of the means - see part (1) of the exercise below - which contains only 30 values.) Denote the column value of each reading by x. To simplify the arithmetic, the true value of the hypothetical quantity is assumed to be zero. Then the error in each reading is equal to x and is thus an integer.

The assumption that the true value is zero is reasonable only if there is no systematic bias in the throwing. To check this, calculate the value of $\langle x \rangle$, the mean value of x for the 150 throws. If $\langle x \rangle$ does not lie between $-\frac{1}{2}$ and $\frac{1}{2}$, renumber the columns, taking as the zero column the one closest to $\langle x \rangle$. All subsequent calculations are done on the basis of the new numbering. This way of reducing the bias keeps the arithmetic simple and is usually adequate for the purpose. The exercise is in three parts.

(1) *Relation between σ and σ_m.* The object of this part is to help you understand the significance of quantities σ and σ_m. After you have thrown all 150 darts count the number of throws n_i in each column x_i. Calculate the standard deviation σ of the distribution from the relation

$$\sigma^2 = \frac{1}{150} \sum n_i x_i^2.$$

Then take the mean $\bar{x}$ of each set of 5 readings. As with the single readings the value of each mean is also its error. The 30 means constitute a distribution of means. Calculate the standard deviation σ_m of this distribution, either by feeding the numbers into a programmed calculator, or from the formula

$$\sigma_m^2 = \frac{1}{30} \sum_j (\bar{x}_j)^2$$

where $\bar{x}_j$ is the mean of the jth set of 5 readings. Compare the ratio σ/σ_m with its theoretical value $\sqrt{5}$.

(2) *Properties of the Gaussian and its integral function.* Plot a histogram for all 150 readings, showing the number of times each reading occurs. Use your value of σ and the values of $f(z)$ given in Appendix A to plot a Gaussian distribution, superposing it on the histogram. (With a programmable calculator you can obtain the required Gaussian directly.)

Use the values of $\phi(z)$ given in Appendix A to plot a graph of the theoretical number of readings that lie within $\pm x$ for a Gaussian distribution, and insert your own values on it. (Think carefully about the x coordinates of the latter.)

(3) *Range method of estimating σ.* Calculate the range r for each of the 30 sets. Find the mean value of r, and use the relation $\sigma \approx r/\sqrt{5}$, with $r = \bar{r}$, to obtain a second estimate of σ. Compare it with the value obtained in part (1).

Calculate the standard deviation ρ of the 30 values of r, and hence the fractional standard deviation $\rho/\bar{r}$. The theoretical expression for the fractional standard deviation of the σ values, when they are estimated by the standard method for each set of 5 readings, is $1/(2n-2)^{\frac{1}{2}}$ - see section 3.7. Compare your value of $\rho/\bar{r}$ with the value of the theoretical expression for $n = 5$, which will indicate the comparative reliabilities of the range and standard methods of estimating σ.

Dr Winter at the Cavendish Laboratory has produced a computer version of the dartboard exercise for the BBC microcomputer. It has the advantage that the values of the simulated measurements (obtained by stopping a moving spot on the monitor screen) are stored in the computer, and hence the arithmetical manipulations can be carried out rapidly. Thus it is easy to carry out the calculations for a range of n values - not just $n = 5$. A listing of the program and a cassette tape are available on application.*

3.2 In an undergraduate practical class in the Cavendish Laboratory there was an experiment, originally devised by Searle, to measure the Young modulus E for steel by applying a known load to a rod and measuring the deflection by an optical method based on Newton's rings. Although ingenious and capable of considerable precision in the hands of a skilled experimenter, such as Searle himself, the results obtained by the students were found to have a considerable scatter. The experiment was therefore replaced by one in which a horizontal steel beam was supported near its ends, and the

* Dr A. T. Winter, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE. There is a small charge for the cassette tape to cover the cost.

deflection when a known load was applied at the centre was measured directly by a dial indicator.

The values obtained for E by the last 10 students who did the Newton's rings experiment and by the first 10 who did the dial indicator experiment are given below. The values are in units of 10^{11} N m^{-2}.

Newton's rings experiment 1·90, 2·28, 1·74, 2·27, 1·67, 2·01, 1·60, 2·18, 2·18, 2·00.

Dial indicator experiment 2·01, 2·05, 2·03, 2·07, 2·04, 2·02, 2·09, 2·09, 2·04, 2·03.

For each set of values, calculate the mean value of E, and estimate the standard error in the mean, by the two methods given in this chapter. Do the results indicate any systematic difference in the two experimental methods?

3.3 The thermal conductivity of copper at 0 °C is

$$k = 385{\cdot}0 \text{ W m}^{-1} \text{ K}^{-1}.$$

A large number of measurements of k, free from systematic error, form a Gaussian distribution with standard error

$$\sigma = 15{\cdot}0 \text{ W m}^{-1} \text{ K}^{-1}.$$

What is the probability that a single measurement lies in the range
(a) 385·0 to 385·1, (b) 400·0 to 400·1, (c) 415·0 to 415·1,
(d) 370·0 to 400·0, (e) 355·0 to 415·0, (f) 340·0 to 430·0 W m^{-1} K^{-1}?

4

Further topics in statistical theory

4.1 The treatment of functions

In most experiments we do not measure the final quantity Z directly. Instead we measure certain primary quantities A, B, C, etc. and calculate Z, which must be a known function of the primary quantities. For example, we might measure the density d of the material of a rectangular block by measuring the mass M and the dimensions l_x, l_y, l_z of the block. The functional relation between the quantity we require, d, and the primary quantities M, l_x, l_y, l_z is

$$d = \frac{M}{l_x l_y l_z}. \tag{4.1}$$

Suppose that each primary quantity has been measured several times. Then, in the case of A, we have a best value $\bar{A}$, the mean of the measured values, and an estimate of its standard error ΔA. (The latter is the σ_m of the previous chapter.) Similarly we have $\bar{B}$ and an estimate of ΔB. We assume that the measurements of the primary quantities are independent and therefore that the errors in them are also independent. By this we mean, for example, that if the value of $\bar{A}$ happens to be high, the value of $\bar{B}$ still has an equal probability of being high or low. From the values $A = \bar{A}$, $B = \bar{B}$, etc., the best value of Z may be calculated. The problem we are going to consider is how to calculate the standard error ΔZ in Z from the standard errors ΔA, ΔB, etc. Although we are restricting the discussion to independent measurements, there are occasional situations where the assumption is not valid. The way ΔZ is calculated in such cases depends on the way the primary errors are related; no general rule can be given. Exercise 4.2 provides an example of related errors.

(*a*) ***Functions of one variable.*** We consider first the case where Z is a function of only one variable A, for example

$$Z = A^2 \quad \text{or} \quad Z = \ln A.$$

We write this in general as

$$Z = Z(A). \tag{4.2}$$

(The symbol A is used both for the name of the primary quantity and for its value.)

If the true value of the primary quantity is A_0, the true value of Z is

$$Z_0 = Z(A_0) \tag{4.3}$$

- see Fig. 4.1. The error in a given value A is

$$E_A = A - A_0, \tag{4.4}$$

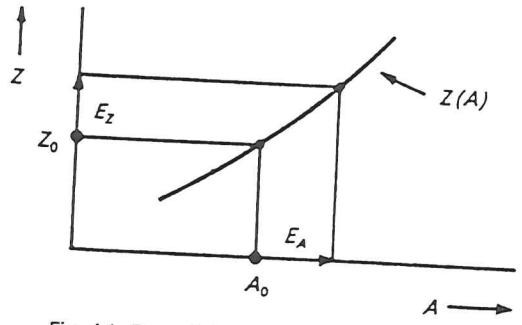

Fig. 4.1. Error E_z in Z due to error E_A in A.

and this gives rise to an error E_Z in Z, where

$$E_Z = Z(A_0 + E_A) - Z(A_0) \tag{4.5}$$

$$\approx \frac{dZ}{dA} E_A. \tag{4.6}$$

The derivative dZ/dA is evaluated at $A = A_0$. The approximation in (4.6) is equivalent to the assumption that the error in A is sufficiently small for $Z(A)$ to be represented by a straight line over the range of the measured values of A. The error in Z is therefore proportional to the error in A, the constant of proportionality being

$$c_A = \left(\frac{dZ}{dA}\right)_{A=A_0}. \tag{4.7}$$

We now allow A to vary according to the distribution of which $\bar{A}$ is a member and take the root-mean-square of (4.6). This gives the result

$$\Delta Z = c_A \Delta A. \tag{4.8}$$

An important special case is $Z = A^n$, for which $c_A = nA^{n-1}$. Then

$$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}, \tag{4.9}$$

i.e. the fractional standard error in Z is n times that in A. We have already used this result (p. 26) for the case $n = \frac{1}{2}$.

(*b*) ***Functions of several variables.*** We next consider the case where Z is a function of two variables A and B,

$$Z = Z(A, B). \tag{4.10}$$

The errors in A and B are

$$E_A = A - A_0, \qquad E_B = B - B_0, \tag{4.11}$$

where A_0 and B_0 are the true values of A and B. As before we assume that Z is approximately a linear function of A and B in the range over which the measured values vary. Then the error in Z is

$$E_Z = c_A E_A + c_B E_B, \tag{4.12}$$

where the coefficients c_A and c_B are given by

$$c_A = \frac{\partial Z}{\partial A}, \qquad c_B = \frac{\partial Z}{\partial B}. \tag{4.13}$$

The partial derivatives are evaluated at $A = A_0$, $B = B_0$.

From (4.12)

$$E_Z^2 = c_A^2 E_A^2 + c_B^2 E_B^2 + 2c_A c_B E_A E_B. \tag{4.14}$$

We take the average of this equation for pairs of values of A and B taken from their respective distributions. Since A and B are assumed independent, the average value of $E_A E_B$ is zero. By definition

$$(\Delta Z)^2 = \langle E_Z^2 \rangle, \qquad (\Delta A)^2 = \langle E_A^2 \rangle, \qquad (\Delta B)^2 = \langle E_B^2 \rangle. \tag{4.15}$$

Therefore

$$(\Delta Z)^2 = c_A^2 (\Delta A)^2 + c_B^2 (\Delta B)^2. \tag{4.16}$$

We can now state the general rule. Let Z be a known function of A, B, C, ... Let the standard error in A be ΔA and so on. Then the standard error ΔZ in Z is given by

$$(\Delta Z)^2 = (\Delta Z_A)^2 + (\Delta Z_B)^2 + (\Delta Z_C)^2 + \ldots, \tag{4.17}$$

where

$$\Delta Z_A = \left(\frac{\partial Z}{\partial A}\right) \Delta A \quad \text{and so on.} \tag{4.18}$$

The expressions for ΔZ for some common relations between Z and A, B are given in Table 4.1.

Table 4.1. *Combination of errors*

Relation between Z and A, B	Relation between standard errors	
$Z=A+B$, $Z=A-B$	$(\Delta Z)^2=(\Delta A)^2+(\Delta B)^2$	(i)
$Z=AB$, $Z=A/B$	$\left(\frac{\Delta Z}{Z}\right)^2=\left(\frac{\Delta A}{A}\right)^2+\left(\frac{\Delta B}{B}\right)^2$	(ii)
$Z=A^n$	$\frac{\Delta Z}{Z}=n\frac{\Delta A}{A}$	(iii)
$Z=\ln A$	$\Delta Z=\frac{\Delta A}{A}$	(iv)
$Z=\exp A$	$\frac{\Delta Z}{Z}=\Delta A$	(v)

4.2 The straight line

In an experiment it is often the case that one quantity y is a function of another quantity x, and measurements are made of pairs of values of x and y. The values are then plotted on a graph and we try to find a curve corresponding to some algebraic function $y=y(x)$ which passes as closely as possible through the points. We shall only consider the case where the function is the straight line

$$y=mx+c. \tag{4.19}$$

The problem is to calculate the values of the parameters m and c for the best straight line through the points.

The straight-line relation covers a great range of physical situations. In fact we usually try to plot the graph so that the expected relationship *is* a straight line. For example, if we expect the refractive index μ of a certain glass to be related to the wavelength λ of the light by the equation

$$\mu=a+b/\lambda^2, \tag{4.20}$$

we plot μ against $1/\lambda^2$.

We give two methods for calculating the best, i.e. most probable, line through a set of points.

(*a*) ***The method of least squares.*** This is the standard statistical method. Suppose there are n pairs of measurements $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

– Fig. 4.2. Assume that the errors are entirely in the y values.* For a given pair of values for m and c, the deviation of the ith reading is

$$y_i-mx_i-c. \tag{4.21}$$

The best values of m and c are taken to be those for which

$$S=\sum(y_i-mx_i-c)^2 \tag{4.22}$$

is a minimum† – hence the name *method of least squares.*

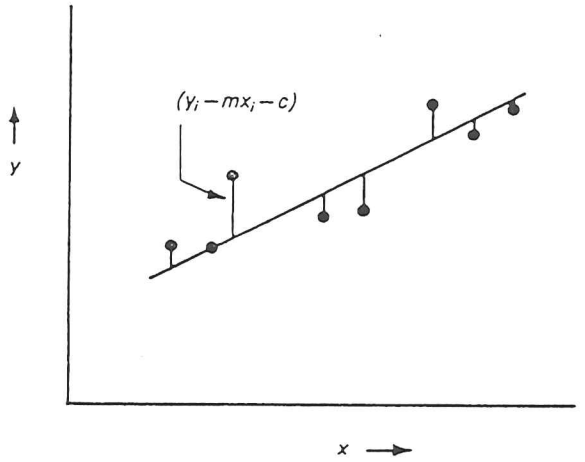

Fig. 4.2. Method of least squares. The best line through the points is taken to be the one for which $\sum(y_i-mx_i-c)^2$ is a minimum.

The principle of minimizing the sum of the squares of the deviations was first suggested by Legendre in 1806. We have already seen that in the case of a single observable the principle gives the mean as the best value.

From (4.22)

$$\frac{\partial S}{\partial m}=-2\sum x_i(y_i-mx_i-c)=0, \tag{4.23}$$

$$\frac{\partial S}{\partial c}=-2\sum(y_i-mx_i-c)=0. \tag{4.24}$$

Therefore the required values of m and c are obtained from the two

* The analysis for the case when there are errors in both the x and y variables is much more complicated – see Guest 1961, p. 128 – but the resulting straight line is usually quite close to that given by the present calculation – see exercise 4.4.

† The points are assumed to have equal weights. The case of unequal weights is discussed in the next section.

simultaneous equations

$$m\sum x_i^2 + c\sum x_i = \sum x_i y_i, \tag{4.25}$$

$$m\sum x_i + cn = \sum y_i. \tag{4.26}$$

The last equation shows that the best line goes through the point

$$\bar{x} = \frac{1}{n}\sum x_i, \qquad \bar{y} = \frac{1}{n}\sum y_i, \tag{4.27}$$

i.e. through the centre of gravity of all the points. From (4.25) and (4.26)

$$m = \frac{\sum (x_i - \bar{x})y_i}{\sum (x_i - \bar{x})^2}, \tag{4.28}$$

$$c = \bar{y} - m\bar{x}. \tag{4.29}$$

When the best values of m and c are inserted into (4.21), the deviations become the residuals

$$d_i = y_i - mx_i - c. \tag{4.30}$$

Estimates of the standard errors in m and c are given by

$$(\Delta m)^2 \approx \frac{1}{D}\frac{\sum d_i^2}{n-2}, \tag{4.31}$$

$$(\Delta c)^2 \approx \left(\frac{1}{n} + \frac{\bar{x}^2}{D}\right)\frac{\sum d_i^2}{n-2}, \tag{4.32}$$

$$D = \sum (x_i - \bar{x})^2. \tag{4.33}$$

These results are proved in Appendix E.

If we require the best line that passes through the origin, the value of m is given by (4.25) with $c = 0$:

$$m = \frac{\sum x_i y_i}{\sum x_i^2}. \tag{4.34}$$

An estimate of its standard error is given by

$$(\Delta m)^2 \approx \frac{1}{\sum x_i^2}\frac{\sum d_i^2}{n-1}. \tag{4.35}$$

(*b*) ***Points in pairs.*** With a computer it is a simple matter to evaluate the best values of m and c from (4.25) and (4.26), and the standard errors from (4.31) and (4.32). With a programmable calculator the evaluation of m and c is still straightforward, but the evaluation of the standard errors is laborious due to the need to calculate the deviations d_i. However, there are other methods that can be used in the absence of a computer. The following is a simple one that is often adequate for the purpose. It is particularly useful when the x values are equally spaced.

In order to illustrate the method, suppose that we have 8 points that lie approximately on a straight line, and we require the best value of the slope m and the error in it. Let the points be numbered in order from 1 to 8 - see Fig. 4.3. Consider points 1 and 5; they determine a straight line and hence a value for the slope. Pairing the points in this way we obtain four values for the slope. We take their mean $\bar{m}$ as the best value of m and find its standard error in the usual way.

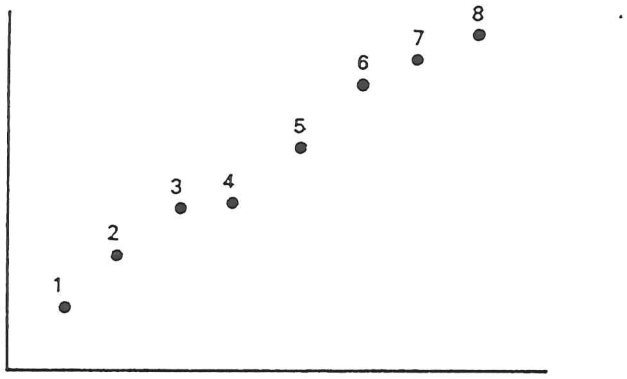

Fig. 4.3. Simple method of estimating slope of best line. Each pair of points 1–5, 2–6, etc. gives a value of the slope. The mean is taken as the best value.

The method will give a reasonable result only if the quantities $(x_5 - x_1)$, $(x_6 - x_2)$, $(x_7 - x_3)$, $(x_8 - x_4)$ are roughly equal. Otherwise, the four values of the slope do not have equal weight.

The best line given by this method is the one with slope $\bar{m}$ that passes through the point $\bar{x}, \bar{y}$. (We have already seen that the line given by the method of least squares passes through this point.) However, the method is mainly used when only the slope is required.

4.3 Weighting of results

Suppose we measure a quantity a certain number of times, say 10, and obtain the values $x_1, x_2, \ldots, x_{10}$. Suppose further that we divide the measurements into two sets and calculate the mean of each. For example, we might make 7 of the measurements in the morning and calculate their mean

$$z_1 = \tfrac{1}{7}(x_1 + x_2 + \ldots + x_7). \tag{4.36}$$

Then we might make the other 3 in the afternoon and calculate their mean

$$z_2 = \tfrac{1}{3}(x_8 + x_9 + x_{10}). \tag{4.37}$$

The best value from all 10 measurements is

$$\bar{z} = \tfrac{1}{10}(x_1 + x_2 + \ldots + x_{10}), \tag{4.38}$$

and obviously it is not given by taking the simple mean of z_1 and z_2. If we wish to calculate it from these two quantities, it is given by

$$\bar{z} = \frac{7z_1 + 3z_2}{10}. \tag{4.39}$$

The numbers 7 and 3 are termed the *weights* or *relative weights* of the quantities z_1 and z_2.

In general, if we have a set of N values $z_1, z_2, \ldots, z_N$ with relative weights $w_1, w_2, \ldots, w_N$, then the best value of the quantity is

$$\bar{z} = \frac{\sum w_i z_i}{\sum w_i}. \tag{4.40}$$

If all the ws are multiplied by a constant, the value of $\bar{z}$ is unchanged, so it is only the ratios of the ws that matter.

Suppose now that we have N measurements of the quantity z, each measurement having its own standard error, i.e. we have

$$z_1 \pm \Delta z_1,\ z_2 \pm \Delta z_2,\ \ldots,\ z_N \pm \Delta z_N.$$

What weight should we give to each z_i in order to obtain the best value of z from all the measurements? The answer is provided by the simple example at the beginning of the section. We saw that if z_i is the mean of n_i original values, then its weight w_i is proportional to n_i. This assumes that all the original values have the same weight, that is, that they all come from the same distribution characterized by a certain standard error σ. We therefore *imagine* each z_i in the above set is the mean of n_i original values taken from a distribution of standard error σ, and give it weight n_i.

We do not know the value of σ; in fact we can choose it quite arbitrarily but, having fixed on a value, we use the result

$$\Delta z_i = \frac{\sigma}{\sqrt{n_i}} \tag{4.41}$$

to obtain n_i. So

$$w_i = n_i = \frac{\sigma^2}{(\Delta z_i)^2}. \tag{4.42}$$

The standard error in $\bar{z}$ is $\sigma/(\sum n_i)^{\frac{1}{2}}$.

From (4.40) and (4.42) the best value of $\bar{z}$ and its standard error are

$$\frac{\sum(1/\Delta z_i)^2 z_i}{\sum(1/\Delta z_i)^2} \pm \frac{1}{[\sum(1/\Delta z_i)^2]^{\frac{1}{2}}}. \tag{4.43}$$

Both these expressions are independent of the value of σ as of course they must be.

In section 4.2 we gave the method of least squares for finding the best straight line through a set of points of equal weight. The generalization to the case of unequal weights is readily made. If w_i is the weight of the pair of values x_i, y_i, then it is necessary to minimize the quantity

$$S_w = \sum w_i(y_i - mx_i - c)^2. \tag{4.44}$$

The equations for m and c become

$$m\sum w_i x_i^2 + c\sum w_i x_i = \sum w_i x_i y_i, \tag{4.45}$$

$$m\sum w_i x_i + c\sum w_i = \sum w_i y_i. \tag{4.46}$$

The expressions for m and c and their standard errors are given in the summary that follows.

Summary of equations for the best straight line by the method of least squares

n points x_i, y_i

Equal weights

General line $y = mx + c$

$$m = \frac{1}{D}\sum(x_i - \bar{x})y_i \qquad (\Delta m)^2 \approx \frac{1}{D}\frac{\sum d_i^2}{n-2}$$

$$c = \bar{y} - m\bar{x} \qquad (\Delta c)^2 \approx \left(\frac{1}{n} + \frac{\bar{x}^2}{D}\right)\frac{\sum d_i^2}{n-2}$$

$$\bar{x} = \frac{1}{n}\sum x_i \qquad \bar{y} = \frac{1}{n}\sum y_i$$

$$D = \sum(x_i - \bar{x})^2 \qquad d_i = y_i - mx_i - c$$

Line through origin $y = mx$

$$m = \frac{\sum x_i y_i}{\sum x_i^2} \qquad (\Delta m)^2 \approx \frac{1}{\sum x_i^2}\frac{\sum d_i^2}{n-1}$$

$$d_i = y_i - mx_i$$

Unequal weights

General line $y = mx + c$

$$m = \frac{1}{D_w}\sum w_i(x_i - \bar{x})y_i \qquad (\Delta m)^2 \approx \frac{1}{D_w}\frac{\sum w_i d_i^2}{n-2}$$

$$c = \bar{y} - m\bar{x} \qquad (\Delta c)^2 \approx \left(\frac{1}{\sum w_i} + \frac{\bar{x}^2}{D_w}\right)\frac{\sum w_i d_i^2}{n-2}$$

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \qquad \bar{y} = \frac{\sum w_i y_i}{\sum w_i}$$

$$D_w = \sum w_i(x_i - \bar{x})^2 \qquad d_i = y_i - mx_i - c$$

Line through origin $y = mx$

$$m = \frac{\sum w_i x_i y_i}{\sum w_i x_i^2} \qquad (\Delta m)^2 \approx \frac{1}{\sum w_i x_i^2}\frac{\sum w_i d_i^2}{n-1}$$

$$d_i = y_i - mx_i$$

Exercises

4.1 In the following examples, Z is a given function of the independently measured quantities $A, B, \ldots$ Calculate the value of Z and its standard error ΔZ from the given values of $A \pm \Delta A$, $B \pm \Delta B, \ldots$

(a) $Z = A^2$, $A = 25 \pm 1$.

(b) $Z = A - 2B$, $A = 100 \pm 3$, $B = 45 \pm 2$.

(c) $Z = \frac{A}{B}(C^2 + D^{\frac{1}{2}})$, $A = 0{\cdot}100 \pm 0{\cdot}003$, $B = 1{\cdot}00 \pm 0{\cdot}05$, $C = 50{\cdot}0 \pm 0{\cdot}5$, $D = 100 \pm 8$.

(d) $Z = A \ln B$, $A = 10{\cdot}00 \pm 0{\cdot}06$, $B = 100 \pm 2$.

(e) $Z = 1 - \frac{1}{A}$, $A = 50 \pm 2$.

4.2 The volume V of a rectangular block is determined by measuring the lengths l_x, l_y, l_z of its sides. From the scatter of the measurements a standard error of 0·01% is assigned to each dimension. What is the standard error in V (a) if the scatter is due to errors in setting and reading the measuring instrument and (b) if it is due to temperature fluctuations?

4.3 A weight W is suspended from the centre of a steel bar which is supported at its ends, and the deflection at the centre is measured by means of a dial height-indicator whose readings are denoted by y. The following values are obtained:

W/kg	y/μm
0	1642
$\frac{1}{2}$	1483
1	1300
$1\frac{1}{2}$	1140
2	948
$2\frac{1}{2}$	781
3	590
$3\frac{1}{2}$	426
4	263
$4\frac{1}{2}$	77

(a) Plot the points on a graph and draw the best line by eye. Make an intelligent guess of the standard error in the slope by placing a transparent rule along the points and seeing what might be reasonable limits for the line.

(b) Calculate the best value of the slope and its standard error by the method of least squares, and compare the results with your estimates in (a).

(c) Calculate the best value of the slope and its standard error by the method of points in pairs, and draw the line with this slope through the point $\bar{x}, \bar{y}$. Compare these results with those of (b).

4.4 The zener diode is a semiconductor device with the property that its resistance drops suddenly to almost zero when the reverse bias voltage exceeds a critical value V_z, which depends on the temperature T of the diode. The value of V_z is of the order of volts, but the temperature coefficient dV_z/dT is only a few millivolts per °C in the temperature range 20–80 °C. Therefore, to measure dV_z/dT precisely, a constant reference voltage is subtracted from V_z, and the resulting voltage V is measured directly on a digital multimeter. The following values are obtained for a particular zener diode:

T/°C	V/mV	T/°C	V/mV
24·0	72·5	50·0	139
30·0	93	56·2	156·5
37·6	107	61·0	171
40·0	116	64·6	178
44·1	127	73·0	198·5

Treat the data in the same way as in parts (a), (b), and (c) of exercise 4.3, assuming in part (b) that the temperature measurements are free from error. This assumption is probably not correct, so repeat the least squares calculation, assuming that the voltage measurements are free from error, and compare the two values of dV_z/dT.

4.5 The results of the 6 most precise measurements of the mass of the charged π meson given in Wohl *et al.* 1984 are

	Mass of $\pi^{\pm}$/keV	
Year	Value	Standard error
1973	139 569	8
1976	139 571	10
1976	139 568·6	2·0
1976	139 566·7	2·4
1979	139 565·8	1·8
1980	139 567·5	0·9

Calculate the weighted mean and its standard error. (The mass m of a nuclear particle is usually given, as here, in terms of its energy equivalent mc^2.)

5

Common sense in errors

5.1 Error calculations in practice

We are now in a position to estimate the standard errors for a large class of experiments. Let us briefly recapitulate. The final quantity Z is a function of the primary quantities A, B, C, ... which are either measured directly or are the slopes or intercepts of straight lines drawn through points representing directly measured quantities.

If the quantity is measured directly, we take the mean of several values to be the best value and obtain its standard error by one of the methods given in chapter 3. (During the present chapter we shall drop the word 'standard' in 'standard error'. We shall not be considering the *actual* error in a measured quantity, and the word 'error' will refer to the standard error, i.e., the standard deviation of the distribution of which the quantity is a member.) If the quantity is the slope or intercept in a straight line, its value and error are obtained either from the method of least squares or from the method of taking the points in pairs.

The best value of Z is calculated from the best values of the primary quantities, and its error is obtained from their errors by the rules given in Table 4.1, or in general from (4.17) and (4.18).

There are often a large number of primary quantities to be measured, and it might be thought that the calculation of the error in each one and the subsequent calculation of the error in Z would be a laborious process. And with many students it is indeed. They calculate the standard deviation automatically for every set of measurements, and combine all the errors irrespective of their magnitudes according to the formal rules, involving themselves in elaborate calculations and ending up with an error calculated to a meaningless number of decimal places, which is usually wrong by several orders of magnitude due to various arithmetical slips on the way.

To see what is required in practice, let us first remember *why* we estimate errors. It is to provide a measure of the significance of the final result. The use made of the error is seldom based on such precise

calculation that we need its value to better than 1 part in 4. Often we are interested in the error to much less precision, perhaps only to within a factor of 2. However, let us take 1 part in 4 as an arbitrary but adequate degree of precision for the final error.

(*a*) ***Combining errors.*** If we look at the equation for combining errors (4.17), we see that, owing to the procedure of squaring the terms, one error is often negligible compared with another. Consider the case

$$Z = A + B, \tag{5.1}$$

and let $\Delta A = 2$ and $\Delta B = 1$. From Table 4.1, (i)

$$\Delta Z = (2^2 + 1^2)^{\frac{1}{2}} = 2{\cdot}24. \tag{5.2}$$

So even though ΔB is as much as one-half of ΔA, ignoring ΔB altogether and putting $\Delta Z \approx \Delta A = 2$ makes a difference of only about 1 part in 8 in the final error. If Z is the sum of several quantities, ignoring errors that are one-half of the largest error may be rather drastic, but we shall nearly always be justified in ignoring an error less than one-third of the largest error.

We may notice also the situation when the quantities themselves differ greatly in magnitude. For example, suppose in (5.1) that B is some small correction term and we have values

$$A = 100 \pm 6$$
$$B = \quad 5 \pm ?$$

The error in B will be negligible unless it is as much as 3, but such an error amounts to 60% of B; so the quantity will have to be measured very roughly indeed if its error is to contribute.

In the case of multiplication and division - Table 4.1, (ii) - we add the squares, not of the errors themselves, but of the fractional errors. So in this case, all fractional errors less than about one-third of the largest fractional error may be neglected.

(*b*) ***Contributing and non-contributing errors.*** With these considerations in mind let us go back to the estimation of the errors in the primary quantities. We may call a quantity *contributing* or *non-contributing* according to whether or not its error contributes appreciably to the final error. A quantity may be non-contributing either because it is measured relatively precisely or because it is added to a much larger quantity.

If we suspect that a quantity is non-contributing, it is sufficient to estimate its error very roughly, provided the estimate is on the high side. The reason for this condition is obvious. It ensures that we do not omit

the error unjustifiably. If the inflated error is negligible we are quite safe. If not, we must go back to the measurements and work out the error more carefully.

For example, suppose the results of successive weighings of an object are:

50·3853 g
50·3846
50·3847
50·3849.

We take the best value of the weight to be

$$50{\cdot}3849 \pm 0{\cdot}0003 \text{ g}.$$

We expect this set of measurements to be much more precise than several others in the particular experiment and we therefore estimate an error simply by inspecting the measurements. The value 0·0003 encompasses 3 out of the 4 individual readings, so it is almost certain to be an overestimate of the error in the mean.

(*c*) ***Discrete readings.*** Another case where a common-sense estimate of the error should be made is when the readings are in digital form or are taken to the nearest scale division of an instrument, and show little or no spread. Consider the following set of measurements made with a metre rule:

$$325, 325, 325, 325\tfrac{1}{2}, 325, 325 \text{ mm}.$$

The most one cay say is that the measured quantity is $325 \pm \frac{1}{2}$ mm or $325 \pm \frac{1}{4}$ mm.* If a better value of the quantity and its error are required, they will not be obtained by more arithmetic, nor by more measurements of the same kind. Either the scale should be estimated to $\frac{1}{10}$ mm as the measurements are made, or a more precise instrument such as a cathetometer should be used.

(*d*) ***Systematic errors.*** So far we have confined the discussion to the estimation of random errors. And this is all that is needed in the majority of experiments. Any systematic error that we know about should be corrected and hence eliminated - or at least rendered negligible. Normally we would reduce it to a level small compared with the random errors. So it would be non-contributing and would not enter the error calculation.

* It is not unknown for students to solemnly feed these numbers into their calculators, arriving at the result $325{\cdot}08 \pm 0{\cdot}08$ mm.

The occasional situation when residual systematic errors are not small compared with random errors should be discussed and treated on its merits. One way of proceeding is to try to estimate, for each systematic error, something equivalent to a standard error, that is to say, a quantity such that we think there are about 2 chances in 3 that the true value lies within the quoted range. For example, we might estimate - or make an intelligent guess of - an upper limit, and then divide it by 2. (This may seem rough and ready, but a crude estimate is better than none at all.) All the errors are then combined as though they were random and independent. When this is done, it is good practice to make quite clear how much of the final error is due to the actual random error and how much to the various systematic errors.

(*e*) ***The final quoted error.*** We may sum up as follows. Systematic errors are eliminated as far as possible. The random errors in contributing quantities are calculated by an appropriate statistical method. Other errors are estimated roughly, the estimates being slightly on the high side. A check - which can often be done mentally - is made that these errors are in fact negligible. The contributing errors are then combined according to the rules of Table 4.1 to give the final quoted error. This quantity represents our best estimate of the standard deviation of the distribution of results that would be obtained if the entire experiment were repeated many times with the same or similar apparatus. It is thus a measure of the overall *reproducibility* of the result.

Some experimenters, having obtained the overall error in the usual way, then proceed to enlarge it by an arbitrary factor to take account of possible, but unknown, sources of systematic error. This is highly undesirable. It is difficult for anyone to make use of these subjective overestimates. You should estimate the error as honestly as you are able and leave it at that. If it subsequently turns out that the 'true' value of the quantity being measured is several times your estimated error away from the value you have obtained, you may or may not be held at fault. But you must not arbitrarily double or treble the error as a kind of safety measure to prevent the situation arising. Quite apart from the confusion caused by the uncertain significance of the final error, the procedure may obscure genuine discrepancies between one experimental result and another, or between theory and experiment.

It is conventional to quote the final error in absolute terms and not as a fraction or percentage. The final value of the quantity being measured and its error should be given to the same number of digits, which should not be more than are meaningful. In general this corresponds to an error of one significant digit, though, if this digit is 1 or 2 a second digit might be given. The fact that we do not want an estimate of the final error more precise than this means that the whole error calculation should be done only to one or at the most two significant digits.

5.2 Complicated functions

The evaluation of quantities of the type $\partial Z/\partial A$ in (4.18) is sometimes quite laborious. As an example consider the measurement of the refractive index μ of a glass prism by measuring the angle A of the prism and the angle D of minimum deviation. The refractive index is obtained from the equation

$$\mu = \frac{\sin \frac{1}{2}(A+D)}{\sin \frac{1}{2}A}. \tag{5.3}$$

The error in μ is given by

$$(\Delta\mu)^2 = (\Delta\mu_A)^2 + (\Delta\mu_D)^2. \tag{5.4}$$

$\Delta\mu_A$ is the error in μ due to the error ΔA in A and is given by

$$\Delta\mu_A = \left(\frac{\partial\mu}{\partial A}\right)\Delta A. \tag{5.5}$$

Similarly for $\Delta\mu_D$.

The expressions for $\partial\mu/\partial A$ and $\partial\mu/\partial D$ are

$$\frac{\partial\mu}{\partial A} = \frac{1}{2}\frac{\cos\frac{1}{2}(A+D)}{\sin\frac{1}{2}A} - \frac{1}{2}\frac{\sin\frac{1}{2}(A+D)}{\sin\frac{1}{2}A\tan\frac{1}{2}A}, \tag{5.6}$$

$$\frac{\partial\mu}{\partial D} = \frac{1}{2}\frac{\cos\frac{1}{2}(A+D)}{\sin\frac{1}{2}A}, \tag{5.7}$$

These expressions have to be evaluated at $A = \bar{A}$, the measured value of A, and $D = \bar{D}$, the measured value of D. And, provided we do the arithmetic correctly and remember to express ΔA and ΔD in radians, we shall get the right answer for $\Delta\mu_A$ and $\Delta\mu_D$.

However, there is a quicker method. Consider the significance of $\Delta\mu_A$. It is the change in the value of μ when A changes by an amount ΔA, the value of D remaining constant. So it may be obtained by calculating μ from (5.3), first for $A = \bar{A}$, $D = \bar{D}$ and then for $A = \bar{A} + \Delta A$, $D = \bar{D}$. The difference is $\Delta\mu_A$. Similarly $\Delta\mu_D$ is obtained by calculating μ for $A = \bar{A}$, $D = \bar{D} + \Delta D$. All we need are the sine values. We do not have to do any complicated algebra or arithmetic - fruitful sources of mistakes - nor bother to convert ΔA and ΔD into radians. We combine $\Delta\mu_A$ and $\Delta\mu_D$ in the usual way.

The fact that this method of calculating $\Delta\mu_A$ and $\Delta\mu_D$ is much quicker than the more formal method should not lead you to imagine that it is in any way less rigorous or exact. The two methods usually give the same answer, and when they do not, the formal method is not valid. This may be seen from Fig. 5.1, where the results of the two methods are shown for some relation $Z = Z(A)$. The best value of A is $\bar{A}$ and this corresponds to Z_1. The error ΔZ_t obtained by the formal method corresponds to putting the tangent to the curve $Z(A)$ at the point $\bar{A}$, Z_1. The value calculated from the simpler and more direct method is ΔZ_+ in the diagram. We could equally well have calculated ΔZ by taking the value $\bar{A} - \Delta A$, which would have given ΔZ_-.

The curvature of the function $Z(A)$ over the range $\bar{A} + \Delta A$ is not usually as large as that shown in Fig. 5.1; in which case the difference between ΔZ_t, ΔZ_+ and ΔZ_- is negligible. If, however, the curvature of the function is significant, then a single value of the error is misleading. Instead, both ΔZ_+ and ΔZ_- should be calculated and the result quoted as

$$Z = Z_1 \begin{matrix} +\Delta Z_+ \\ -\Delta Z_- \end{matrix}.$$

Such refinement is seldom justified. The main point is that to calculate the error in Z due to an error in A, we cannot go wrong if we simply

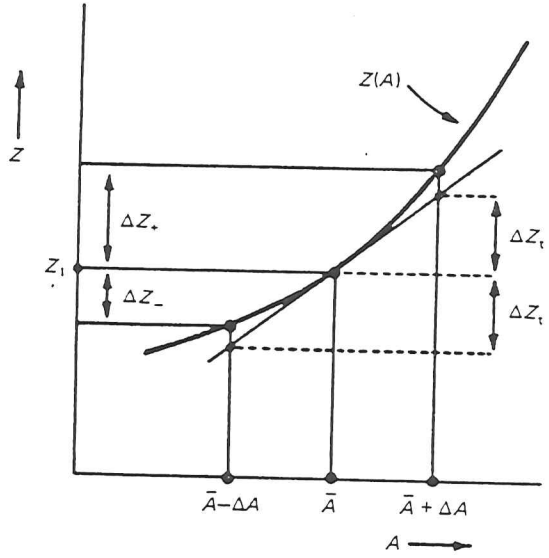

Fig. 5.1. Relation between different estimates of ΔZ

calculate the values of Z at the values $\bar{A}$ and $\bar{A} + \Delta A$, with the other measured quantities constant. And often this is much quicker than the formal method.

5.3 Errors and experimental procedure

When the final quantity Z is related to two directly measured quantities by a function of the form

$$Z = AB \quad \text{or} \quad A/B,$$

then an error of $x\%$ in A or B gives rise to an error of $x\%$ in Z. So we would try to measure A and B with comparable precision, and this is true whatever the relative magnitudes of A and B. But the situation

$$Z = A + B \quad \text{or} \quad A - B$$

is quite different. Everything depends on the relative magnitudes of A and B. Look at the following example:

Case I
$$A = 10\,000 \pm 1,$$
$$B = \quad 100 \pm 5,$$
$$Z = A + B = 10\,100 \pm 5.$$

Here A is a large, precisely known quantity. B has been measured to 5%, but the final quantity Z has been found to 0·05%. So we see that it is advantageous to start with a large, precisely known quantity and simply measure a small additional term in order to get the required quantity.

Now consider the following:

Case II
$$A = 100 \pm 2,$$
$$B = \quad 96 \pm 2,$$
$$Z = A - B = \quad 4 \pm 3.$$

The two directly measured quantities have been determined to 2%, but the final quantity is only known to 75%. So taking the difference between two nearly equal quantities, each of which is measured independently, is inherently disadvantageous; the final error is greatly magnified. If possible an entirely different method of measuring Z should be found.

In the next two chapters we shall give specific examples of methods devised to take advantage of the Case I situation and others devised to avoid Case II. They provide examples of the way error considerations may have a direct influence on experimental procedure.

We give one hypothetical example here. Consider the following situation. We require to measure the quantity $Z = A/B$. We have made a set

of measurements and found

$$A = 1000 \pm 20,$$
$$B = \quad 10 \pm 1.$$

Therefore

$$\frac{\Delta A}{A} = 2\% \quad \text{and} \quad \frac{\Delta B}{B} = 10\%,$$

whence

$$\frac{\Delta Z}{Z} = (2^2 + 10^2)^{\frac{1}{2}} = 10{\cdot}2\%.$$

We have some further time available for measurements and estimate it is sufficient to reduce the error in either A or B by a factor 2. If we devote the time to A, we shall have

$$\frac{\Delta A}{A} = 1\%, \quad \text{which gives} \quad \frac{\Delta Z}{Z} = (1^2 + 10^2)^{\frac{1}{2}} = 10{\cdot}0\%.$$

If we devote it to B, we shall have

$$\frac{\Delta B}{B} = 5\%, \quad \text{which gives} \quad \frac{\Delta Z}{Z} = (2^2 + 5^2)^{\frac{1}{2}} = 5{\cdot}4\%.$$

So in the first case the overall error is barely changed, and in the second case it is reduced by a factor of almost 2. The moral is *always concentrate on quantities that contribute most to the final error.*

In general one should plan the experiment so that in the final result no one quantity contributes an error much greater than the others. In the present example we may suspect that the original measurements, which resulted in $\Delta B/B$ being 5 times greater than $\Delta A/A$, were badly planned, and that more time should have been devoted to measuring B at the expense of A. Of course it is not always the case that additional measurements result in a reduction of the error. Nevertheless, the desirability of reducing the maximum contributing error should always be kept in mind when planning an experiment.

Exercises

5.1 A rectangular brass bar of mass M has dimensions a, b, c. The moment of inertia I about an axis in the centre of the ab face and perpendicular to it is

$$I = \frac{M}{12}(a^2 + b^2).$$

The following measurements are made:

$$M = 135{\cdot}0 \pm 0{\cdot}1 \text{ g},$$
$$a = 80 \pm 1 \text{ mm},$$
$$b = 10 \pm 1 \text{ mm},$$
$$c = 20{\cdot}00 \pm 0{\cdot}01 \text{ mm}.$$

What is the percentage standard error in (a) the density ρ of the material, and (b) the moment of inertia?

5.2 When a torsion wire of radius r and length l is fixed at one end and subjected to a couple of moment C at the other, the angular displacement ϕ is given by

$$\phi = \frac{2lC}{n\pi r^4},$$

where n is the rigidity modulus of the material of the wire. The following values are obtained:

$$\phi/C = 4{\cdot}00 \pm 0{\cdot}12 \text{ rad N}^{-1}\text{ m}^{-1},$$
$$r = 1{\cdot}00 \pm 0{\cdot}02 \text{ mm},$$
$$l = 500 \pm 1 \text{ mm}.$$

Calculate the value of n and its standard error.

5.3 If a narrow collimated beam of monoenergetic γ-rays of intensity I_0 is incident on a thin sheet of material of thickness x, the intensity of the emerging beam is given by

$$I = I_0 \exp(-\mu x),$$

where μ is a quantity known as the linear attenuation coefficient. The following values are obtained for γ-rays of energy 1 MeV incident on lead:

$$I = (0{\cdot}926 \pm 0{\cdot}010) \times 10^{10}\,\gamma\text{-rays m}^{-2}\text{ s}^{-1},$$
$$I_0 = (2{\cdot}026 \pm 0{\cdot}012) \times 10^{10}\,\gamma\text{-rays m}^{-2}\text{ s}^{-1},$$
$$x = (10{\cdot}00 \pm 0{\cdot}02) \text{ mm}.$$

Calculate the value of μ and its standard error for γ-rays of this energy in lead.

5.4 Neutrons reflected by a crystal obey Bragg's law $n\lambda = 2d \sin\theta$, where λ is the de Broglie wavelength of the neutrons, d is the spacing between the reflecting planes of atoms in the crystal, θ is the angle between the incident (or reflected) neutrons and the atomic planes, and n is an integer. If n and d are known, the measured value of θ for a beam of monoenergetic neutrons determines λ, and hence the kinetic energy E of the neutrons. If $\theta = 11°18' \pm 9'$, what is the fractional error in E?

5.5 The frequency f of a tuning fork is related to the length L of its arms and the value of the Young modulus E of its material by

$$f \propto \sqrt{(EL)}.$$

When the temperature rises by 10 K, the frequency of a certain fork falls

by $(0{\cdot}250 \pm 0{\cdot}002)\%$. For the same temperature rise, the Young modulus of the material falls by $(0{\cdot}520 \pm 0{\cdot}003)\%$. Calculate the value of α, the coefficient of linear expansion of the material, given by these experiments. What is its standard error? Is this a good method for measuring the coefficient of linear expansion?

PART 2

Experimental methods